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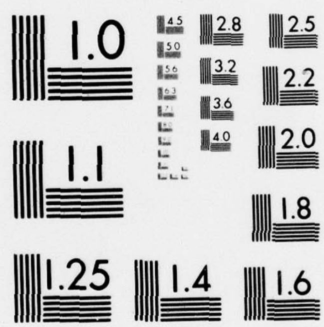

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Optimal Filters For Nilpotent Associate-Algebraic Bilinear Systems *

by

10 Shirish D. Chikte [redacted] James T. Lo **

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Abstract

We consider a bilinear signal process driven by a, Gauss-Markov process which is observed in additive, white, gaussian noise. An exact stochastic differential equation for the least squares filter is derived when the signal process satisfies a nilpotency condition. It is shown that the filter is also bilinear and moreover that it satisfies an analogous nilpotency condition. Finally, an example is presented and ways of reducing filter dimensionality discussed.

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OPTIMAL FILTERS FOR NILPOTENT ASSOCIATE-ALGEBRAIC BILINEAR SYSTEMS

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1. INTRODUCTION

Recently, optimal estimation and detection of signal processes generated by bilinear dynamical systems has been the subject of investigation in a number of articles, e.g. [1], [2] and [3]. Problems of this type arise in such practical applications as inertial navigation, satellite attitude control and angle modulation.

We consider least-squares filtering of stochastic processes generated by the so called nilpotent bilinear dynamical system driven by a Gauss-Markov process observed in white gaussian noise environment. In view of the work by Fliess [4] and Sussman [5], there is strong reason to suspect that the above class of systems can approximate a much wider class of nonlinear systems. The existence of finite dimensional, recursive filters for such processes was established in [3] under a somewhat weaker Lie-algebraic nilpotency condition. In the present paper, explicit stochastic differential equations realizing such filters are derived by "closing" the infinite dimensional nonlinear filtering equations of Kushner [6]. It is found that the filter structure is not only bilinear but it also inherits the nilpotency property from the original system.

The next section presents the problem formulation. The third section contains the derivation of filter equations. The final section concludes with an example and computational considerations.

2. PROBLEM STATEMENT

We first delineate the class of nonlinear filtering problems addressed to in this paper.

We are given the standard linear Itô models for the signal process $\{\xi(t)\}_{t \geq 0}$ and the observation process $\{z(t)\}_{t>0}$ respectively, as follows.

The signal model:

$$d\xi(t) = F(t)\xi(t)dt + Q^{\frac{1}{2}}(t)dw(t)\ , \quad t \geq 0 \tag{2.1}$$

The observation model:

$$dz(t) = H(t)\xi(t)dt + R^{\frac{1}{2}}(t)dv(t), \quad t > 0 \tag{2.2}$$

where $w(\cdot)$ and $v(\cdot)$ are standard N and P dimensional independent Wiener processes respectively, $\xi(t) \in \mathbb{R}^N$, $z(t) \in \mathbb{R}^P$, $\xi(0)$ is a zero mean gaussian random vector independent of $w(\cdot)$ and $v(\cdot)$ processes and $F(\cdot)$, $Q^{\frac{1}{2}}(\cdot)$, $H(\cdot)$, $R^{\frac{1}{2}}(\cdot)$ are time-dependent matrices of appropriate dimensions with $Q(t)$, $R(t)$ positive definite and continuously differentiable for all t.

Now consider a process $\{x(t)\}_{t \geq 0}$ generated by a bilinear dynamical system of the following form:

$$dx(t) = (A + \sum_{i=1}^{N} B_i \xi_i(t))x(t)dt, \quad x(t) \in \mathbb{R}^M, \ t \geq 0 \tag{2.3}$$

where $x(0)$ is a gaussian random vector independent of $\xi(0)$, $\{w(t)\}_{t \geq 0}$, $\{v(t)\}_{t \geq 0}$.

We are interested in an exact, finite dimensional system of stochastic differential equations for the least squares estimator $E[x(t)|z^t] \triangleq \hat{x}(t|t)$ of $x(t)$. It is well-known (see e.g. [6]) that this, in general, is not possible. In this paper, we intend to derive such a filter and study its characteristics with the proviso that the matrices A and $\{B_i\}_{i=1}^{N}$ in (2.3) satisfy a well-known algebraic property called nilpotency.

Definition 1:

An associative algebra $\mathcal{A}$ of matrices is said to be nilpotent (of order K) if there exists a positive integer K such that the (matrix) product of any K or more matrices in $\mathcal{A}$ vanishes.

Assumption 2:

The associative algebra $\mathcal{A}$ generated by the set of matrices $\{Ad_A^k(B_i) \mid i=1,2,\ldots,N;\ k=0,1,\ldots\}$ is nilpotent (of order K, say) where

$$Ad_A^0(B_i) \triangleq B_i$$

and

$$Ad_A^k(B_i) \triangleq [A, Ad_A^{k-1}(B_i)] \triangleq A \cdot Ad_A^{k-1}(B_i) - Ad_A^{k-1}(B_i) \cdot A$$

for $k = 1,2,\ldots$

Remark:

Assumption 2 is stronger than the Lie-algebraic nilpotency assumption made in [3].

3. THE STRUCTURE OF THE NONLINEAR FILTER

The filtering problem posed in the last section, was shown to be amenable to solution in [3]. In the main theorem of this paper that follows shortly, we explicitly exhibit the structure of the nonlinear filter and study its characteristics. We begin with some lemmas.

Lemma 3:

The input-output map realized by the system (2.3) satisfying Assumption 2 can also be realized by a bilinear system which is a "direct sum" of (a finite number of) finite dimensional systems of the following form

$$d\bar{x}^k(t) = (\bar{A}^k + \sum_{i=1}^{N} \bar{B}_i^k \xi_i(t))\, \bar{x}^k(t)dt, \quad t \geq 0 \tag{3.1}$$

where,

$$\bar{A}^k = \begin{bmatrix} A_1^k & 0 & \cdots & \cdots & 0 \\ 0 & A_2^k & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & A_{M_k}^k \end{bmatrix} \tag{3.2}$$

with A_i^k, $i=1, \ldots, M_k$ being $J_i \times J_i$ Jordan blocks, and

$$\bar{B}_i^k = \begin{bmatrix} 0 & B_{i,1}^k & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & B_{i,2}^k & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ \vdots & & & & 0 & B_{i,M_k-1}^k \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix} \tag{3.3}$$

with $B_{i,j}^k$, $i=1,\ldots,N$; $j=1,\ldots,M_k-1$ being $J_j \times J_{j+1}$ matrices.

Hence, the associative algebra $\left\{ Ad_{\bar{A}}^k (\bar{B}_i) \mid k=0,1,\ldots;\ i=1,\ldots,N \right\}_{AA}$ is also nilpotent of the same order K.

Proof:

The proof follows closely the ideas in the proofs of theorems 4 and 6 of Brockett [7] and hence is kept brief.

Using the transformation $z(t) = e^{-At} x(t)$ and applying the Peano-Baker formula, we find that the Volterra series solution to (2.3) is given by

$$x(t) = [e^{At} + \int_0^t e^{A(t-\sigma_1)} \Big(\sum_{i=1}^N B_i \xi_i(\sigma_1) \Big) e^{A\sigma_1} d\sigma_1$$

$$+ \int_0^t \int_0^{\sigma_1} e^{A(t-\sigma_1)} \Big(\sum_{i=1}^N B_i \xi_i(\sigma_1) \Big) \cdot e^{A(\sigma_1 - \sigma_2)}$$

$$\Big(\sum_{j=1}^N B_j \xi_j(\sigma_2) \Big) e^{A\sigma_2} d\sigma_1 d\sigma_2 + \ldots\ldots\ldots]\ x(0) . \tag{3.4}$$

It now follows from the Baker-Campbell-Hausdorff formula together with Assumption 2 that all the terms in (3.4) beyond the first K terms will vanish.

Now, let the Jordan form of A be given by $\operatorname{diag}\left\{A_1, A_2, \ldots, A_s\right\}$ where A_i is, say, $J_i \times J_i$. Let $K_i \triangleq \sum_{k=1}^{i-1} J_k$, $i=1,\ldots,s$ with

$K_1 \triangleq 0$, and define $y^i \triangleq [x_{K_i+1} \cdots\cdots x_{K_{i+1}}]^T$, $i=1,\ldots,s$. Recall that e^{At} consists of $J_i X J_k$ blocks, $i,k=1,\ldots,s$ which can be written as $\sum_{\ell=1}^{s} C_\ell e^{A_\ell t} D_\ell$, for some $J_i X J_\ell$ matrices C_ℓ and $J_\ell X J_k$ matrices D_ℓ.

With these remarks in mind, it is not difficult to see that the r^{th} order term, $1 \leq r < K$, in the Volterra series (3.4), for the subset $y^i(\sigma_o)$ of components of $x(\sigma_o)$, $i=1,\ldots,s$, consists of groups of terms of the form

$$\int_0^{\sigma_o}\int_0^{\sigma_1}\cdots\int_0^{\sigma_{r-1}} \sum_{m_1,\ldots,m_r=1}^{N} C_{1}e^{A_1(\sigma_o-\sigma_1)} D_1 e^{A_2(\sigma_1-\sigma_2)} D_2 \cdots D_r e^{A_r\sigma_r} y^k(0) \prod_{i=1}^{r} \xi_{m_i}(\sigma_i)d\sigma_i$$

But each group of this type is realizable by a system of the type claimed in the lemma with $\bar{A}$ and $\{\bar{B}_i\}_{i=1}^N$ as in (3.2), (3.3) (see theorem 4 of [7] for details). The final assertion regarding nilpotency is obvious if we note that $\bar{A}$ is upper triangular while $\bar{B}_i$, $i=1,\ldots,N$ are strictly upper triangular.

<u>Lemma 4</u>:

Consider the signal and observation models of (2.1) and (2.2) respectively. Define a (vector) process $\{y(t)\}_{t \geq 0}$ by

$$dy(t) = Dy(t)dt + \sum_{i=1}^{N} E_i \xi_i(t)y(t)$$

where $D, \{E_i\}_{i=1}^N$ are matrices of appropriate dimension,

$y(0)$ is independent of $\xi(0)$, $w(\cdot)$ and $v(\cdot)$ processes. (3.5)

Then $\hat{y}(t|t) \triangleq E[y(t)/z^t]$ satisfies the following stochastic differential equation:

$$d\hat{y}(t|t) = D\hat{y}(t|t)dt + \sum_{i=1}^{N} E_i \cdot E^t[\xi_i(t)y(t)]dt$$
$$+ \left\{E^t[y(t)\xi^T(t)] - \hat{y}(t|t)\hat{\xi}^T(t|t)\right\} H^T(t)R^{-1}(t)d\nu(t)$$

$$\hat{y}(0|0) = E[y(0)] \tag{3.6}$$

where,

$$E^t[\cdot] \triangleq E[\cdot|z^t] \triangleq E[\cdot|\{z(\tau) \mid 0 \leq \tau \leq t\}]$$

$$d\nu(t) = dz(t) - H(t)\hat{\xi}(t|t)dt .$$

Proof:

Apply the Kushner nonlinear filtering equations [6] to the signal process $(y^T(\cdot),\xi^T(\cdot))^T$ with $z(\cdot)$ as the observation process.

Lemma 5:

Let $x = (x_0,x_1,\ldots,x_n)^T$ be a gaussian random vector with mean vector $m = (m_0,m_1,\ldots,m_n)^T$ and covariance matrix $P = [P_{ij}]_{i\cdot j = 0}^{n}$. We then have the following relation:

$$E[\prod_{i=0}^{n} x_i] = \begin{cases} m_0 E[\prod_{i=1}^{n} x_i] + \sum_{j=1}^{n} P_{0j} E[\prod_{\substack{i=2 \\ i\neq j}}^{n} x_i], & n > 1 \\ m_0 m_1 + P_{01}, & n = 1. \end{cases} \tag{3.7}$$

Proof: See e.g. [8].

We are now ready to state and prove the main theorem of this paper.

Theorem 6:

Consider the signal process $\{x(t),\xi(t)\}_{t\geq 0}$ evolving as in (2.3) and (2.1) and the observation process $\{z(t)\}_{t\geq 0}$ as in (2.2). Assume that (2.3) satisfies Assumption 2. Then the least squares filtered estimate $\hat{x}(t|t)$ can be obtained via the following finite dimensional system of bilinear stochastic differential equations with suitable initial conditions:

$$d\hat{x}^*(t|t) = (A^*(t)dt + \sum_{i=1}^{N} B_i^* \hat{\xi}_i(t|t)dt + \sum_{i=1}^{N} C_i^* d\mu_i(t))\hat{x}^*(t|t)dt$$

$$\hat{x}(t|t) = L\hat{x}^*(t|t), \quad t \geq 0 \tag{3.8}$$

where $\hat{\xi}(t|t)$, $t \geq 0$ is given by the standard Kalman-Bucy filter [9], $A^*(\cdot)$ is a deterministic matrix time function. L, $\{B_i^*,C_i^*\}_{i=1}^{N}$ are constant matrices, and

$$\mu(t) \triangleq [\mu_1(t)\ \mu_2(t) \cdots \mu_N(t)]^T \triangleq \int_0^t H^T(\tau)\, R^{-1}(\tau)d\nu(\tau)$$

is the modified innovations process.

Furthermore, the associative algebra generated by the following set of matrices,

$$\left\{ Ad^k_{(Ad^{\ell}_{C_i^*}(A^*(t)))}(B_j^*) \,\middle|\, i,j=1,2,\ldots,N;k,\ell=0,1,2,\ldots \right\} \quad t \geq 0$$

is nilpotent of order K.

Proof:

From Lemma 3, it is clear that we may take - without loss of generality - (2.3) to be a stationary bilinear system in which the matrices A and $\{B_i\}_{i=1}^{N}$ are of the special form stipulated in (3.2) and (3.3) respectively. Furthermore, we shall assume for ease of exposition, that

$A = \text{diag}(a_1, a_2, \ldots, a_M)$; consequently $B_k \triangleq [b_k^{i,j}]$ is such that

$$b_k^{i,j} = \begin{cases} 0, & j \neq i+1 \\ b_2^k, & j = i+1 \end{cases}, \quad k=1,\ldots,N \quad \text{and} \quad x_i(0) = 0 \quad \text{a.s.} \quad \text{for } i < M. \quad \text{For,}$$

it would become apparent in the sequel, the proof for the general case would involve little more than additional notational complexity.

We now apply Lemma 4 to (2.3) with the above mentioned specialization. With the help of Lemma 5, the resulting differential equation analogous to (3.6) can be written in the following form:

$$d\hat{x}(t|t) = (A + \sum_{j=1}^{N} B_j \hat{\xi}_j(t|t))\hat{x}(t|t)dt + \sum_{j=1}^{N} B_j \hat{x}^j(t|t)dt$$
$$+ [\hat{x}^1(t|t), \hat{x}^2(t|t), \ldots, \hat{x}^N(t|t)]H^T(t)R^{-1}(t)d\nu(t) \tag{3.9}$$

where the "augmenting states" appearing in (2.17) are defined as

$$x^j(\cdot) \triangleq [x_1^j(\cdot), x_2^j(\cdot), \ldots, x_M^j(\cdot)]^T, \quad j=1,2,\ldots,N,$$

$$x_q^k(\sigma_0) = \begin{cases} \displaystyle\sum_{m_1,\ldots,m_{M-q}=1}^{N} \sum_{r=1}^{M-q} \int_0^{\sigma_0}\int_0^{\sigma_1}\cdots\int_0^{\sigma_{M-q-1}} P_{k,m_r}(\sigma_0,\sigma_r)e^{a_{q+r}(\sigma_r-\sigma_{r+1})} \cdot \\ \quad \cdot b_{q+r}^{m_r} \cdot e^{a_M \sigma_{M-q}} x_M(0) \Big(\prod_{\substack{i=0 \\ i\neq r}}^{M-q-1} e^{a_{q+i}(\sigma_i-\sigma_{i+1})} m_{i+1} \cdot b_{q+i}^{m_{i+1}} \\ \quad \cdot \xi_{m_{i+1}}(\sigma_{i+1}) d\sigma, & q < M \\ 0, & q = M \end{cases} \tag{3.10}$$

(Note: In the general case $x_q^k(\sigma_0)$ would be a vector of a dimension consistent with the q^{th} Jordan block).

and,
$$P_{i,j}(t,\sigma), \quad i,j=1,\ldots,N\ ;\ 0 \leq \sigma \leq t,$$

is the i,j^{th} element of the conditional covariance matrix

$$P(t,\sigma) = E[(\xi(t) - \hat{\xi}(t|t))(\xi(\sigma) - \hat{\xi}(\sigma|t))^T \Big|_{z^t}] \tag{3.11}$$

Also recall [10] that $P(t,\sigma)$, $\sigma \leq t$ satisfies the following differential equation

$$\frac{\partial P(t,\sigma)}{\partial t} = [F(t) - P(t)H^T(t)R^{-1}(t)H(t)]P(t,\sigma). \tag{3.12}$$

Now, if we differentiate the augmenting state vector

$$^1x(\cdot) \triangleq [x^{1^T}(\cdot)\ x^{2^T}(\cdot), \ldots x^{N^T}(\cdot)]^T \in \mathbb{R}^{MN}, \tag{3.13}$$

then utilizing (3.12), we find that

$$d\,{}^1x(t) = \alpha^1(t)\,{}^1x(t)dt + \beta^1(t)x(t)dt + \sum_{i=1}^{N}\gamma_i^1\,{}^1x(t)\xi_i(t)dt \tag{3.14}$$

where $\alpha^1(\cdot)$, $\beta^1(\cdot)$, $\{\gamma_i^1\}_{i=1}^N$ are determined by the problem parameters and the error covariance $P(\cdot)$, the $(M\times M)$ blocks of $\alpha^1(\cdot)$ and $\beta^1(\cdot)$ are upper triangular and $\{\gamma_i^1\}_{i=1}^N$ are block diagonal matrices with identical strictly upper triangular blocks given by

$$\begin{bmatrix} 0 & b_1^i & 0 & - & - & - & 0 \\ 0 & 0 & b_2^i & 0 & & & 0 \\ \vdots & & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & b_{M-2}^i & & 0 \\ \vdots & & & & \ddots & \ddots & 0 \\ 0 & \cdots & & & \cdots & 0 & 0 \end{bmatrix} \triangleq \text{a block of } \gamma_i^1 \tag{3.15}$$

Now we may again apply Lemmas 4 and 5 to (3.14), and get

$$\begin{aligned} d\,{}^1\hat{x}(t|t) &= \Big(\alpha^1(t) + \sum_{i=1}^{N}\gamma_i^1\hat{\xi}_i(t|t)\Big){}^1\hat{x}(t|t)dt + \beta^1(t)\hat{x}(t|t)dt \\ &+ \sum_{j=1}^{N}\gamma_j^1\,{}^1\hat{x}^j(t|t)dt + [{}^1\hat{x}^1(t|t), {}^1\hat{x}^2(t|t),\ldots,{}^1\hat{x}^N(t|t)]H^T(t)R^{-1}(t)d\nu(t) \end{aligned} \tag{3.16}$$

where the "new" set of augmenting states are

$${}^1x^j(\cdot) \triangleq [x^{j,1^T}(\cdot), x^{j,2^T}(\cdot),\ldots\ x^{j,N^T}(\cdot)]^T\ ,\ j=1,\ldots,N$$

$$x^{j,k}(\cdot) \triangleq [x_1^{j,k}(\cdot),\ x_2^{j,k}(\cdot),\ldots,x_M^{j,k}(\cdot)]^T$$

$$x_q^{j,k}(\sigma_o) = \begin{cases} \displaystyle\sum_{m_1,\ldots,m_{M-q}=1}^{N}\ \sum_{\substack{r_1,r_2=1\\ r_1\neq r_2}}^{M-q}\int_0^{\sigma_o}\int_0^{\sigma_1}\cdots\int_0^{\sigma_{M-q-1}} P_{j,m_{r_1}}(\sigma_o,\sigma_{r_1})P_{k,m_{r_2}}\cdot(\sigma_o,\sigma_{r_2}) \\ \qquad e^{a_{q+r_1}(\sigma_{r_1}-\sigma_{r_1+1})}\, b_{q+r_1}^{m_{r_1}}\, e^{a_{q+r_2}(\sigma_{r_2}-\sigma_{r_2+1})}\,\cdot \\ \qquad e^{a_M\sigma_{M-q}}\, x_M(0)\Big(\displaystyle\prod_{\substack{i=0\\ i\neq r_1,r_2}}^{M-q-1} e^{a_{q+i}(\sigma_i-\sigma_{i+1})}\cdot b_{q+i}^{m_{i+1}}\xi_{m_{i+1}}(\sigma_{i+1})d\sigma_{i+1}\Big), \\ \qquad q < M-1 \\ 0 \qquad q \geq M-1 \end{cases} \tag{3.17}$$

Now the new augmenting state vector

$$ {}^2x(\cdot) \triangleq [{}^1x^1(\cdot)^T,\ {}^1x^2(\cdot)^T, \dots, {}^1x^{N^T}]^T \in \mathbb{R}^{MN^2} \tag{3.18} $$

analogous to (3.16), is seen to satisfy the differential equation

$$ d\,{}^2x(t) = \alpha^2(t)\,{}^2x(t)dt + \beta^2(t)\,{}^1x(t)dt + \sum_{i=1}^{N} \gamma_i^2\,{}^2x(t)\xi_i(t)dt \tag{3.19} $$

in which $M \times M$ blocks of $\alpha^2(\cdot)$, $\beta^2(\cdot)$ and γ_i^2 have the same properties as those of $\alpha^1(\cdot)$, $\beta^1(\cdot)$ and $\gamma_i^2(\cdot)$ respectively with a block of γ_i^2 obtained by setting $b^i_{M-2} = 0$ in the block (3.15) of γ_i^1. The above process can now be iterated and it is clear that this sequence will terminate at the M^{th} application - in the general case, this would be the total number of Jordan blocks - of Kushner equations since $\gamma_i^{M-1} = 0$, $i=1,2,\dots,N$. Collecting together the differential equations for $x(\cdot), {}^1x(\cdot), \dots {}^{M-1}x(\cdot)$, we see that the dimension of the resulting nonlinear filter equals $(M+MN+MN^2+\dots+MN^{M-1}) = \frac{M}{N-1}(N^M-1)$. The filter form (3.8) follows upon writing the innovations term in the standard bilinear format, which also reveals the fact that the matrices $\{C_i^*\}_{i=1}^N$ consist of upper triangular $M \times M$ blocks. We thus see that $\{B_i^*\}_{i=1}^N$ consists of (strictly) upper triangular $(M \times M)$ blocks, while $\{A^*(t)\}_{t \ge 0}$ and $\{C_i^*\}_{i=1}^N$ consist of $(M \times M)$ blocks that are upper triangular. It is, therefore, easy to see - if we carry out matrix multiplications *blockwise* - that the set of matrices $\left\{Ad^{\ell}_{C_i^*}(A^*(t)) \,\middle|\, t\ge 0; i=1,\dots,N; \ell=0,1,\dots\right\}$ also have $(M \times M)$ blocks all upper triangular. Consequently the set

$$ \left\{ Ad^k_{(Ad^{\ell}_{C_i^*}(A^*(t)))}(B_j^*) \,\middle|\, t\ge 0; i,j=1,\dots,N; k,\ell=0,1,2,\dots \right\} $$

has $(M \times M)$ blocks that are all strictly upper triangular. Hence, blockwise multiplication shows that the associative algebra generated by them is nilpotent of order K.

Q.E.D.

The filter structure revealed in the above theorem has some interesting features worth noting. We first see that analogous to the case of linear signal models, the "drift" terms in (3.8) do preserve the bilinear structure of the signal model (2.14) and also inherit its nilpotency property; nevertheless the matrix $A^*(\cdot)$ is time varying although the matrix A is constant. Secondly, the filter is bilinear in the innovations as well. Also observe that the state spaces of both the signal model and the filter are (not necessarily identical) nilpotent group manifolds. This suggests that it may be worthwhile to investigate optimal filters under criteria that are defined on such manifolds. Such an approach for abelian groups was followed in [1].

4. COMPUTATIONAL CONSIDERATIONS

Realization of the filter (2.16) in the form of a block schematic is shown in figure 1 on the following page. The practical importance of the bilinear property of the above filter is that a real-time analog implementation of the filter is still possible with easily available and cheap hardware consisting of integrators, summers and multipliers.

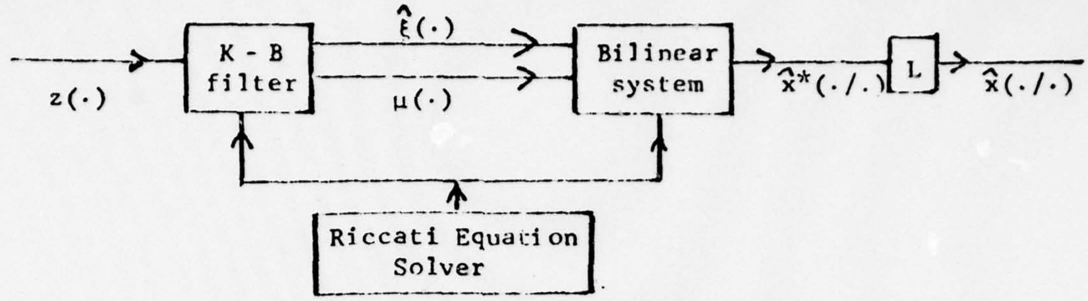

Figure 1

However, a major drawback of the filter is its "curse of dimensionality". The following example illustrates the problem and shows one way of its mitigation. Example: We now apply the algorithm developed in the theorem to construct the optimal filter for the case of system (2.3) with $M = 3$,

$$N = 2 \text{ and } A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 & b_1^i & 0 \\ 0 & 0 & b_2^i \\ 0 & 0 & 0 \end{bmatrix}, \quad i = 1,2$$

The filter is found to have a dimension of 21, having required 3 applications of Kushner's equations. We enumerate below the nonzero 3×3 blocks of $A^*(t)$, $\{B_k^*\}_{k=1}^N$, $\{C_k^*\}_{k=1}^N$ and L. The notation used is as follows:

$T^{i,j}$: i,j^{th} block of matrix T

$P_{i,j}(t), \epsilon_{i,j}(t)$: i,j^{th} elements of matrices $P(t)$ and $[F(t)-P(t)H^T(t)R^{-1}(t)H(t)]$ respectively.

$\delta(\cdot)$: Krönecker delta function.

and finally, for any $n \times n$ matrix T and $0 \leq i \leq n$,

T^i: Matrix T with last i rows and columns replaced by zeros.

$$A^{*1,1}(t) = A; \; A^{*1,j}(t) = B^1_{j-1}, \; j=2,3;$$

$$A^{*i,1}(t) = \sum_{k=1}^{2} P_{(i-1),k}(t) B_k; \; A^{*i,j}(t) = [A^1 + \epsilon_{(i-1),(j-1)}(t) I_3^1], \quad i,j=2,3;$$

$$A^{*4,2}(t) = 2\sum_{i=1}^{2} B_i^1 P_{1,i}(t), \; A^{*4,j}(t) = 2[A^2 + \epsilon_{1,(j-3)}(t) I_3^2], \; j=4,5;$$

$$A^{*i,2}(t) = \sum_{k=1}^{2} P_{2,k}(t) B_k, \; i=5,6; \; A^{*i,3}(t) = \sum_{k=1}^{2} P_{1,k}(t) B_k, \; i=5,6;$$

$$A^{*i,j}(t) = [A^2 + \epsilon_{2,(j-3)}(t) I_3^2], \; i=5,6; \; j=4,5;$$

$$A^{*i,j}(t) = [A^2 + \epsilon_{1,(j-5)}(t) I_3^2], \; i=5,6; \; j=6,7;$$

$$A^{*7,3}(t) = 2\sum_{k=1}^{2} P_{2,k}(t) B_k^1;$$

$$A^{*7,j}(t) = 2[A^2 + \epsilon_{2,(j-5)}(t) I_3^2] \ , \ j = 6,7$$

$$B_k^{*1,1} = B_k; \ B_k^{*i,1} = B_k, i=2,3; \ k=1,2$$

$$C_k^{*1,2} = \delta(k-1) I_3^1 \ ; \ C_k^{*1,3} = \delta(k-2) \cdot I_3^1 \ ;$$

$$C_k^{*2,4} = \delta(k-1) \cdot I_3^2 \ ; \ C_k^{*2,6} = \delta(k-2) \cdot I_3^2 \ ;$$

$$C_k^{*3,5} = \delta(k-1) \cdot I_3^2 \ ; \ C_k^{*3,7} = \delta(k-2) \ I_3^2 \ ; \ k = 1,2$$

$$L^{1,1} = I_3$$

In the above matrices, note the repetition of certain block rows and the presence of some all zero rows. This allows us in effect to reduce the filter dimension from 21 to 10. This argument can be extended to the general case, and it is not difficult to show that the dimensionality can be reduced in this way from

$$\frac{M}{N-1}(N^M - 1) \quad \text{to} \quad \sum_{i=0}^{M-1} (M-i) \binom{N+i-1}{i}$$

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We consider a bilinear signal process driven by a, Gauss-Markov process which is observed in additive, white, gaussian noise. An exact stochastic differential equation for the least squares filter is derived when the signal process satisfies a nilpotency condition. It is shown that the filter is also bilinear and moreover that it satisfies an analogous nilpotency condition. Finally, an example is presented and ways of reducing filter dimensionality discussed.

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